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Single-Look Detection with Unknown Signal Strengths

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This problem concerns receiving two different waveforms that contain either noise or signal plus noise. The number of signals present is to be determined. The signal amplitudes are unknown and are the unknown parameters of the problem. Classical results involve estimating the unknown parameters and selecting the most likely condition. However, the total probability of error for the detection of the number of signals by using the classical procedure is high. Consequently, several alternate detection procedures are developed. All methods considered give better results than the classical method, although no single method is found to be always best. The effect of correlated noise is also investigated.

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SINGLE-LOOK DETECTION WITH UNKNOWN SIGNAL STRENGTHS

INTRODUCTION

Detection problems involving unknown parameters can lead to large error probabilities. An example problem illustrating this has two signals of unknown strength added to Gaussian noise. Let hypothesis H_n $(n = 0,1,2)$ designate that exactly n signals are present. Mathematically, the problem with equal a priori probabilities is stated as:

$$P(H_o) = P(H_1) = P(H_2) = 1/3$$

$$H_0\colon r_1 = n_1$$

$$r_2 = n_2$$

$$H_1\colon r_1 = s_1 + n_1$$

$$r_2 = n_2$$

or

$$r_1 = n_1$$

$$r_2 = s_2 + n_2$$

$$H_2\colon r_1 = s_1 + n_1$$

$$r_2 = s_2 + n_2,$$

where r_1 and r_2 are received signals, n_1 and n_2 are noises, and s_1 and s_2 are signals of unknown strengths. After receiving one sample each of r_1 and r_2, the decision problem is to determine how many signals are present.

The classical method estimates the signal strengths from the received voltages under each hypothesis, places these signal strengths back into the corresponding pdf (probability density function), and chooses the number of signals present that gives the minimum error probability. This method gives a 2/3 error probabality for any combination of signal strengths. The poor performance is due to having only a single observation and equal a priori probabilities.

A detector based on knowing the signal values provides an upper limit on performance and is the standard to which other methods are compared. The comparison of detector performance is based on the total probability of error. The spread between the classical and the known signal performance suggests that there should be some decision procedures having error probabilities lying between them. The "known signal" rule can be applied, even when the actual signal strengths are unknown, where a 'guess' is made for the strengths. This rule and the one assuming that the signals are uniformly distributed between two limits are applied to this problem. Neyman-Pearson type tests are also developed and applied to this problem. The Neyman-Pearson tests are generalized and applied to correlated noise.

PROBABILITY DENSITY FUNCTIONS

Gaussian noises n_1 and n_2 of zero mean and equal variance are added to the signals s_1 and s_2. For most of the work, n_1 and n_2 are independent. With the received signals normalized to the

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noisestandard deviation, the pdfs for H_0, H_1, and H_2 are given below. H_n is the hypotheses that exactly n signals are present. All parameters of the noise are assumed to be known. The Gaussian noise density is

$$p_{\bar{n}}(n_1, n_2) = \frac{1}{2\pi} e^{-\frac{n_1^2 + n_2^2}{2}},$$

and the probability densities for the received signals under each hypothesis are

$$\begin{aligned} p_{\bar{r}|H_0}(r_1,r_2) &= p_{\bar{n}}(r_1, r_2) \\ &= \frac{1}{2\pi} e^{-\frac{r_1^2 + r_2^2}{2}}, \\ p_{\bar{r}|H_1}(r_1,r_2) &= P(s_1 \neq 0, s_2 = 0)\, p_{\bar{n}}(r_1 - s_1, r_2) + P(s_1 = 0, s_2 \neq 0)\, p_{\bar{n}}(r_1, r_2 - s_2) \\ &= \frac{1}{2} p_{\bar{n}}(r_1 - s_1, r_2) + \frac{1}{2} p_{\bar{n}}(r_1, r_2 - s_2) \\ &= \frac{1}{2}\frac{1}{2\pi} e^{-\frac{(r_1-s_1)^2 + r_2^2}{2}} + \frac{1}{2}\frac{1}{2\pi} e^{-\frac{r_1^2+(r_2-s_2)^2}{2}}, \end{aligned}$$

and

$$\begin{aligned} p_{\bar{r}|H_2}(r_1,r_2) &= p_{\bar{n}}(r_1 - s_1, r_2 - s_2) \\ &= \frac{1}{2\pi} e^{-\frac{(r_1-s_1)^2 + (r_2-s_2)^2}{2}} \end{aligned}$$

The probability density function of jointly Gaussian correlated noise of equal variances and correlation coefficient ρ is given by

$$p_{\bar{n}}(n_1,n_2) = \frac{1}{2\pi(1-\rho^2)^{1/2}} e^{-\frac{(n_1^2 - 2\rho\, n_1 n_2 + n_2^2)}{2(1-\rho^2)}}$$

The probability density of the received signals under each hypothesis is modified accordingly.

MINIMUM PROBABILITY OF ERROR

The decision of whether H_0, H_1, or H_2 has occurred is made after looking at the received signals r_1 and r_2 and using the known a priori probabilities. The minimum probability of error occurs when the probability of being correct is a maximum. The decision rule is

$$\max_{H_i} P(H_i |\text{ observe } \bar{r}) \rightarrow \text{choose } H_i$$

or equivalently,

$$\max_{H_i} P(H_i)\, p_{\bar{r}|H_i}(r_1,r_2) \rightarrow \text{choose } H_i .$$

If $P(H_i)\, p_{\bar{r}|H_i}(r_1,r_2)$ is compared with $P(H_j) p_{\bar{r}|H_j}(r_1,r_2)$, then the hypothesis having the smallest value can never be the one giving the maximum value (at least, it would be rejected in favor of the hypothesis having the larger value in this test). Thus, the result of conducting the maximization operation is equivalent to choosing the hypothesis that satisfies a set of binary tests (expressed in the negative of each hypothesis) [1, pp. 48-50]. For the case of three hypotheses, this set of tests is:

H_0 and H_1: $P(H_0) p_{\bar{r}|H_0}(\bar{r}) < P(H_1)\, p_{\bar{r}|H_1}(\bar{r}) \rightarrow$ not H_0, else not H_1,

H_0 and H_2: $P(H_0) p_{\bar{r}|H_0}(\bar{r}) < P(H_2)\, p_{\bar{r}|H_2}(\bar{r}) \rightarrow$ not H_0, else not H_2, and

H_1 and H_2: $P(H_1) p_{\bar{r}|H_0}(\bar{r}) < P(H_2)\, p_{\bar{r}|H_2}(\bar{r}) \rightarrow$ not H_1, else not H_2.

The hypothesis minimizing the error probability is that which holds true for each of these tests. These tests are applied by the following logic:

```
If test {H0 and H1} is true, then
   If test {H1 and H2} is true, then
      choose hypothesis H2
   else
      choose hypothesis H1
   endif
else
   If test {H0 and H2} is true, then
      choose hypothesis H2
   else
      choose hypothesis H0.
   endif
endif.
```

CLASSICAL METHOD

The classical method estimates the unknown parameters in each of the pdfs and then applies the minimum probability of error test. We shall use the maximum a posteriori (MAP) estimate

$$\max_{\overline{s}} p(\overline{r}|H_n;\overline{s}) \to \overline{s} \text{ for } H_n,\ n = 0,1,2\,.$$

For each hypothesis H_n, the unknown parameters are estimated. In this case, the unknown parameters are the signal levels s_1 and s_2, and the relevant pdfs are

$$H_0\text{: } p(r_1,r_2|H_0) = \frac{1}{2\pi}e^{-1/2(r_1^2 + r_2^2)},$$

$$H_1\text{: } p(r_1,r_2|H_1) = \frac{1}{2\pi}\left\{\frac{1}{2}e^{-1/2\left[(r_1-s_1)^2 + r_2^2\right]} + \frac{1}{2}e^{-1/2\left[r_1^2 + (r_2-s_2)^2\right]}\right\},$$

and

$$H_2\text{: } p(r_1,r_2|H_2) = \frac{1}{2\pi}e^{-1/2\left[(r_1-s_1)^2 + (r_2-s_2)^2\right]}.$$

The estimates for s_1 and s_2 that maximize these pdfs are:

$$H_0 : \text{ (no parameters needed)}\,,$$

$$H_1 : \ s_1 = r_1^* \text{ and } s_2 = r_2^*\,,$$

and

$$H_2 : \ s_1 = r_1^* \text{ and } s_2 = r_2^*\,,$$

where r_1^* and r_s^* are the received values of r_1 and r_2, respectively. The minimum probability of error decision rule is to choose the hypothesis

$$H_0 : \ \frac{1}{3}\,\frac{1}{2\pi}\,e^{-1/2\left[(r_1^*)^2 + (r_2^*)^2\right]},$$

$$H_1 : \ \frac{1}{3}\,\frac{1}{2\pi}\,\frac{e^{-1/2(r_1^*)^2} + e^{-1/2(r_2^*)^2}}{2},$$

and

$$H_2 : \ \frac{1}{3}\,\frac{1}{2\pi}$$

with the largest value. By inspection, H_2 always has a larger probability density function, hence H_2 is always chosen as the final decision. Since H_2 occurs only one out of three opportunities, the error probability is 2/3.

KNOWN-SIGNAL DETECTOR

To place an upper bound on the performance that any method could yield, the minimum probability of error detector was designed where the signal strengths were used in the design. The performance for any other procedure never exceeds the performance obtained when there is perfect knowledge of the signal strengths; i.e., the values of s_1 and s_2 are known.

For equally likely a priori probabilities, $P(H_0) = P(H_1) = P(H_2) = 1/3$.

By letting $R_1 = e^{s_1(r_1 - 1/2s_1)}$ and $R_2 = e^{s_2(r_2 - 1/2s_2)}$, the tests may then be expressed as

$$
\begin{aligned}
&H_0 \text{ and } H_1: && 1 < \frac{R_1 + R_2}{2} && \rightarrow \text{not } H_0, \text{ else not } H_1, \\
&H_0 \text{ and } H_2: && 1 < R_1 R_2 && \rightarrow \text{not } H_0, \text{ else not } H_2, \text{ and} \\
&H_1 \text{ and } H_2: && \frac{R_1 + R_2}{2} < R_1 R_2 && \rightarrow \text{not } H_1, \text{ else not } H_2.
\end{aligned}
$$

Note that R_1 and R_2 are the likelihood ratios of (H_0 and H_1) and (H_0 and H_2), respectively. These tests can be shown graphically on a R_1 and R_2 plot where each (R_1, R_2) pair lies in a decision region (labeled for H_0, H_1, or H_2). The boundaries between the decision regions are obtained by replacing the inequalities by equalities. All decision boundaries are shown in Fig. 1 where parts (a), (b), and (c) show the boundary of each test individually, and (d) shows all boundaries. Note that the H_0, H_2 boundary is never used. Figure 2 shows all regions and boundaries needed to make a decision.

Table 1 shows the total error probability for known signals lying between −16 dB and +16 dB. Figure 3 shows a plot taken along the main diagonal. The small signal asymptote is the error probability that is obtained when nothing is known about the signal; for equally likely a priori probabilities, the small signal asymptote error probability is 2/3. The error probability drops to 0.02 when the signal is 16 dB and approaches zero for large signals.

MINIMAX CRITERION

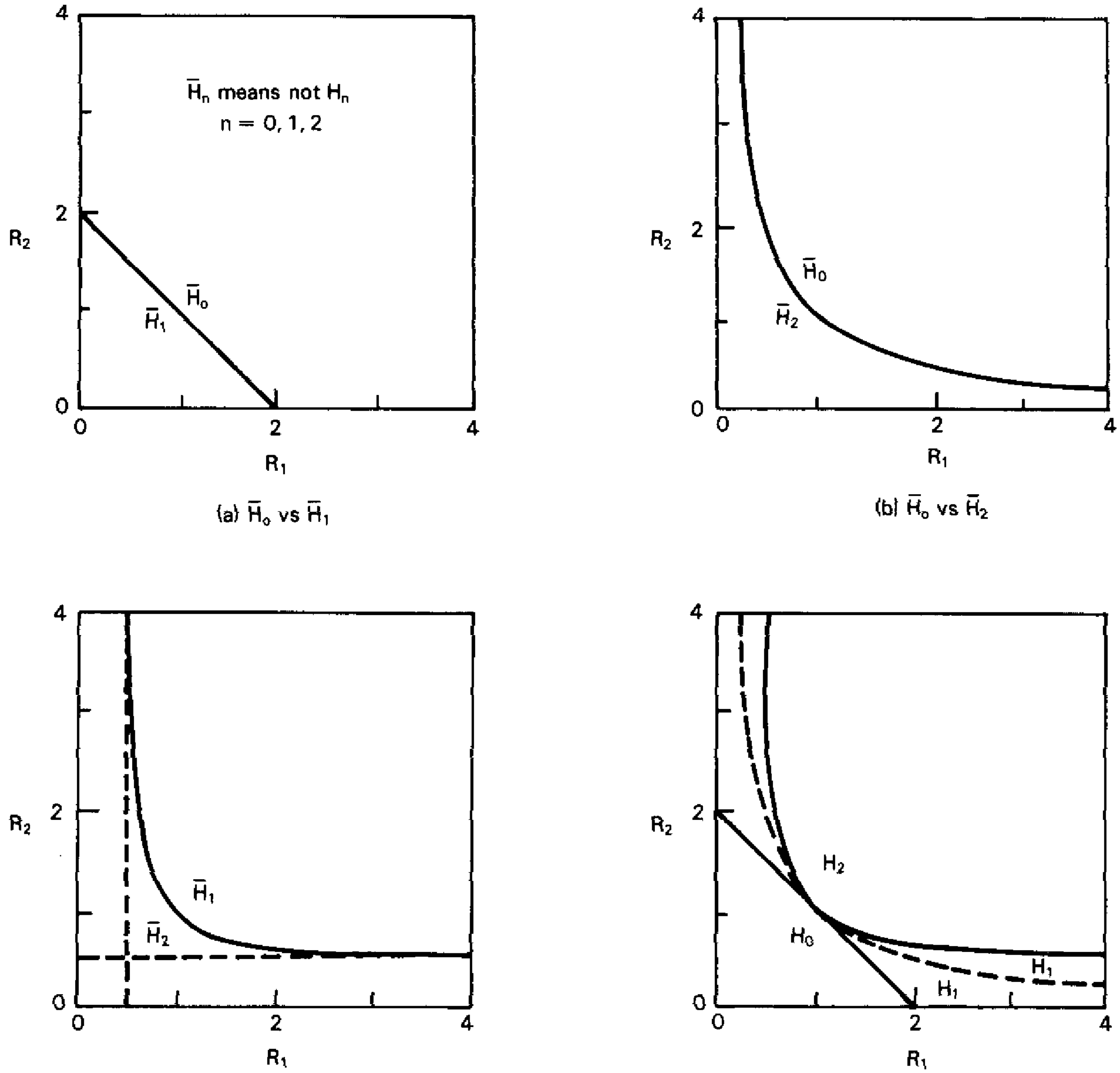
The decision rules considered contain parameters that are at our disposal. To select design parameters $\vec{d}$, a minimax test is conducted. The minimax design selects the design parameters $\vec{d}$ by performing

$$\min_{\vec{d} \in D} \max_{\vec{s}_{actual} \in A} Q(\vec{s}_{actual}, \vec{d}) .$$

The selected design is that $\vec{d}$ which yields this minimax value for Q where Q is a performance measure, $\vec{s}_{actual}$ is the actual signal value, **A** is the set of allowed signal vectors, $\vec{d}$ is the design parameters at our disposal, and **D** is the set of allowed design vectors.

For this work, the performance Q is the difference between the probability of error using the subject design and the probability of error using the true signal. For this performance measure, Q is identified by the more descriptive notation $\Delta P\ (\epsilon)$.

$$\Delta P\ (\epsilon) = P \left\{ \text{error} \mid \begin{matrix}\text{design based on} \\ \text{assumed signals}\end{matrix} \right\} - P \left\{ \text{error} \mid \begin{matrix}\text{design based on} \\ \text{actual signals}\end{matrix} \right\}$$



(a) $\bar{H}_0$ vs $\bar{H}_1$

(b) $\bar{H}_0$ vs $\bar{H}_2$

(c) $\bar{H}_1$ vs $\bar{H}_2$

(d) all boundaries

Fig. 1 — "Known-signal" decision boundaries

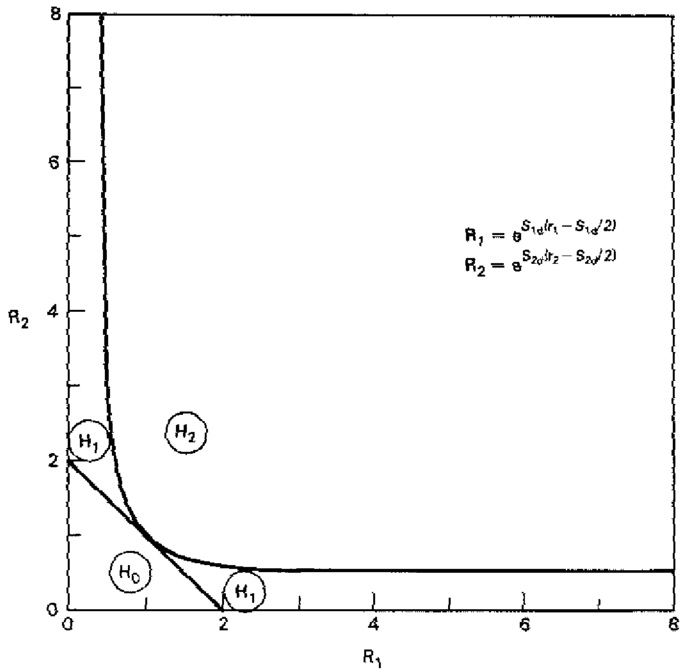


Fig. 2 — "Known-signal" decision regions

Table 1 — $P(\epsilon)$ for "known signal"

S_1 (dB)

S_2 (dB)	-16	-14	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12	14	16
-16	0.6367	0.6305	0.6222	0.6162	0.6017	0.5933	0.5753	0.5573	0.5290	0.5012	0.4658	0.4302	0.3993	0.3700	0.3498	0.3387	0.3347
-14	0.6300	0.6297	0.6213	0.6103	0.6070	0.5897	0.5747	0.5572	0.5283	0.5017	0.4673	0.4305	0.3998	0.3712	0.3500	0.3388	0.3350
-12	0.6220	0.6167	0.6195	0.6078	0.5970	0.5862	0.5720	0.5510	0.5253	0.4975	0.4665	0.4285	0.4000	0.3688	0.3498	0.3385	0.3347
-10	0.6150	0.6088	0.6045	0.6043	0.5948	0.5782	0.5658	0.5442	0.5268	0.4997	0.4663	0.4268	0.3980	0.3687	0.3508	0.3392	0.3360
-8	0.6037	0.5993	0.5942	0.5880	0.5822	0.5688	0.5555	0.5358	0.5208	0.4927	0.4653	0.4250	0.3982	0.3658	0.3485	0.3378	0.3347
-6	0.5935	0.5852	0.5820	0.5773	0.5752	0.5633	0.5457	0.5328	0.5102	0.4790	0.4510	0.4202	0.3892	0.3582	0.3398	0.3313	0.3267
-4	0.5753	0.5712	0.5708	0.5617	0.5517	0.5477	0.5363	0.5167	0.4910	0.4663	0.4380	0.4067	0.3742	0.3478	0.3252	0.3168	0.3128
-2	0.5553	0.5557	0.5510	0.5423	0.5350	0.5282	0.5190	0.4930	0.4728	0.4488	0.4212	0.3930	0.3567	0.3292	0.3087	0.2988	0.2958
0	0.5327	0.5277	0.5240	0.5213	0.5195	0.5100	0.4962	0.4745	0.4532	0.4293	0.3983	0.3675	0.3313	0.3055	0.2835	0.2720	0.2693
2	0.5048	0.5045	0.5030	0.5028	0.4915	0.4858	0.4712	0.4532	0.4280	0.4017	0.3738	0.3343	0.2988	0.2675	0.2465	0.2365	0.2338
4	0.4708	0.4693	0.4690	0.4658	0.4572	0.4570	0.4425	0.4257	0.3993	0.3718	0.3383	0.2975	0.2600	0.2272	0.2053	0.1933	0.1903
6	0.4378	0.4360	0.4372	0.4382	0.4302	0.4283	0.4185	0.3973	0.3665	0.3345	0.3000	0.2547	0.2160	0.1787	0.1575	0.1462	0.1423
8	0.3987	0.3985	0.3978	0.3967	0.3945	0.3895	0.3778	0.3647	0.3335	0.3012	0.2668	0.2180	0.1752	0.1380	0.1147	0.1015	0.0973
10	0.3703	0.3700	0.3707	0.3688	0.3630	0.3605	0.3507	0.3340	0.3050	0.2745	0.2325	0.1853	0.1395	0.0992	0.0743	0.0588	0.0540
12	0.3498	0.3502	0.3505	0.3490	0.3448	0.3417	0.3310	0.3128	0.2820	0.2507	0.2083	0.1572	0.1128	0.0718	0.0455	0.0300	0.0252
14	0.3393	0.3393	0.3397	0.3368	0.3350	0.3323	0.3238	0.3018	0.2708	0.2393	0.1970	0.1443	0.0990	0.0577	0.0302	0.0150	0.0100
16	0.3343	0.3343	0.3347	0.3320	0.3310	0.3283	0.3200	0.2982	0.2665	0.2345	0.1923	0.1397	0.0932	0.0518	0.2835	0.0083	0.0030

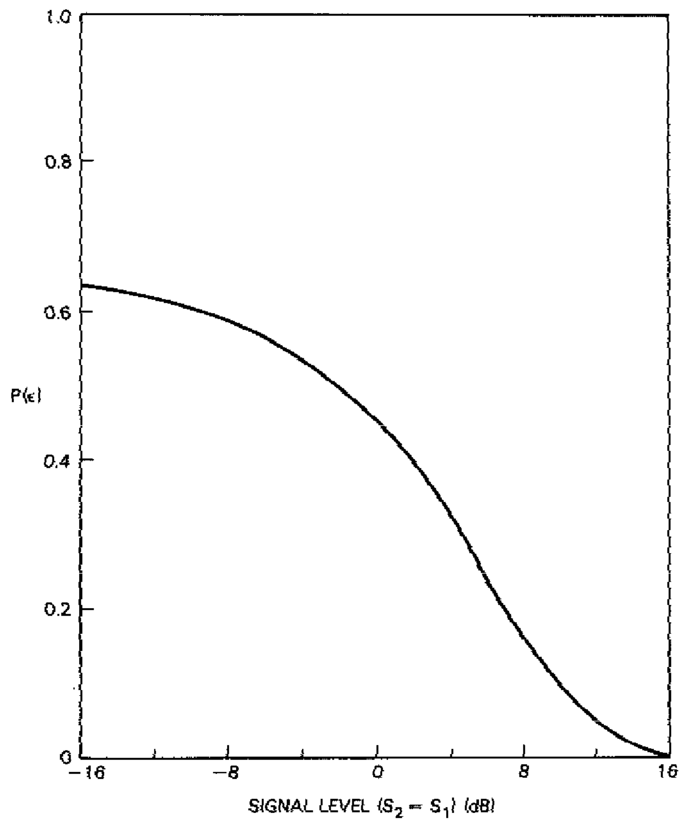


Fig. 3 — Error probability for "known-signal" design (reference)

ASSUMED KNOWN-SIGNAL DECISION RULE

Perfect knowledge of s_1 and s_2 combined with the minimum probability of error criterion leads to the rules for the "known-signal" detector. By assuming values for s_1 and s_2, we obtain the "assumed known-signal" decision rules. The signal parameters in R_1 and R_2 are the design parameters at our disposal. To help pick these values, two criterions are used: minimax and min-average. The min-average lead to the same general results as the minimax and are not reported here.

The problem is solved by considering the following statements: (1) the best design requires $s_{2d} = s_{1d}$; (2) the location of the actual signals yielding the worst performance is at either $s_2 = s_1$ or $s_{2\,max}$ and $s_{1\,min}$ (or $s_{1\,min}$ and $s_{2\,max}$); and (3) the best design occurs at the intersection of the limiting values of two families of curves associated with either $s_2 = s_1$ or s_{max} and s_{min}. We consider each statement independently and then together.

Table 2 contains the worst (maximum) $\Delta P(\epsilon)$ for the design signals s_{1d} and s_{2d} ranging from −16 dB to +16 dB (each entry is the "max" part of "minimax"). Thus, each entry in Table 2 is the result of maximizing 81 $\Delta P(\epsilon)$ values for a table total of 6561 = 81 × 81 calculations. This table shows the best design to have $s_{2d} = s_{1d}$ (this is the "min" part of "minimax"). This table places the best design near $s_{2d} = s_{1d} = 8$ dB. Consequently, we have demonstrated the first statement (1), that the best design lies along the diagonal $s_2 = s_1$.

Statement (2), which claims that the worst performance occurs for signals of $s_2 = s_1$ or when one signal is a maximum and the other is a minimum, is shown in Table 3. Each entry in Table 3 gives the location of s_1 and s_2 which corresponds to the worst $\Delta P(\epsilon)$, where $\Delta P(\epsilon)$ is computed for all values of the actual signals s_1 and s_2. The entries in this table are of the form (m,n) where m and n = signal value (1 = −16 dB, 2 = −12 dB,..., 9 = +16 dB) and $m \Leftrightarrow s_1$. For example, the design of

Table 2 — Worst $\Delta P_r(\epsilon)$ for Each Design Pair

S_1 (dB)

S_2(dB)	-16	-12	-8	-4	0	4	8	12	16
-16	0.4888	0.4793	0.4487	0.4058	0.3470	0.2968	0.2712	0.2843	0.4123
-12	0.4820	0.4802	0.4583	0.4138	0.3565	0.2888	0.2165	0.2262	0.3443
-8	0.4520	0.4613	0.4517	0.4152	0.3597	0.2925	0.2200	0.1865	0.3130
-4	0.4042	0.4165	0.4143	0.3940	0.3393	0.2762	0.2052	0.1918	0.3167
0	0.3505	0.3565	0.3610	0.3445	0.3017	0.2465	0.1820	0.1957	0.3182
4	0.3085	0.2913	0.2920	0.2802	0.2448	0.1963	0.1625	0.1903	0.3110
8	0.2820	0.2242	0.2240	0.2127	0.1837	0.1565	0.1490	0.1772	0.2933
12	0.2887	0.2275	0.1855	0.1872	0.1837	0.1745	0.1672	0.1988	0.2953
16	0.4038	0.3372	0.3030	0.3068	0.3042	0.2978	0.2888	0.2890	0.3958

Table 3 — Location of Worst $\Delta P_r(\epsilon)$ Actual Levels for "Assumed-Known Signal" Detector

S_1 (dB)

S_2(dB)	-16	-12	-8	-4	0	4	8	12	16
-16	9,9	9,9	9,9	9,9	9,9	9,8	9,8	5,7	7,8
-12						9,9	9,9	5,7	7,7
-8								5,9	7,9
-4								5,9	7,5
0	9,9					9,9	9,9	5,9	7,9
4	8,9					9,9	1,9	5,9	7,9
8	8,9	9,9	9,9	9,9	9,9	9,1	**1,9**	5,9	7,9
12	7,6	7,6	9,6	9,6	9,6	9,5	9,1	6,6	7,6
16	8,7	7,7	9,7	9,7	9,7	9,6	9,6	6,7	8,7

$s_{2d} = s_{1d} = 12$ dB (located near the lower right-hand corner of Table 3) has the actual signals giving the largest $\Delta P(\epsilon)$ of (6,6) or $s_1 = s_2 = 4$ dB. Two points can be made. Since the best signal design is $s_{2d} = s_{1d}$, we need consider only what happens there. Two actual signal locations occur on the main diagonal at (max, max) or at (min, max). Both depend on the signal levels used in the computer run. Observe that, except for the design at (8 dB, 8 dB) where the actual signals are minimax, all other designs along the main diagonal have actual signals of near equal strength ($s_2 = s_1$). This observation on Table 3 verifies statement 2.

Statement (3) is considered in three parts: (a) determine the maximum $\Delta P(\epsilon)$ over all possible actual signals $s_2 = s_1$ for each design signal $s_{2d} = s_{1d}$; (b) determine the maximum $\Delta P(\epsilon)$ for all actual signals $(s_{1\,\min}, s_{2\,\max})$ or $(s_{1\,\max}, s_{2\,\min})$ for all design signals $s_{2d} = s_{1d}$; and (c) determine maximum $\Delta P(\epsilon)$ from (a) and (b).

Working on part (a) first, Fig. 4 shows $\Delta P(\epsilon)$ vs $s_{2d} = s_{1d}$ for various values of $s_2 = s_1$. Figure 4 indicates that for any signal design, $\Delta P(\epsilon)$ is nearly the same for all large actual signals. Furthermore, $\Delta P(\epsilon)$ is a maximum at large actual signal values for small designs (left side of curve), and $\Delta P(\epsilon)$ is a maximum at small actual signal designs when the design signals are large. The envelope of the curves defined by the crosshatched area form the curve of maximum $\Delta P(\epsilon)$ vs signal design $s_{2d} = s_{1d}$. Of course, the minimax solution is to choose the signal design associated with the minimum of this envelope curve that occurs at $s_{d2} = s_{d1} \approx 9$ dB, if only part (a) is considered.

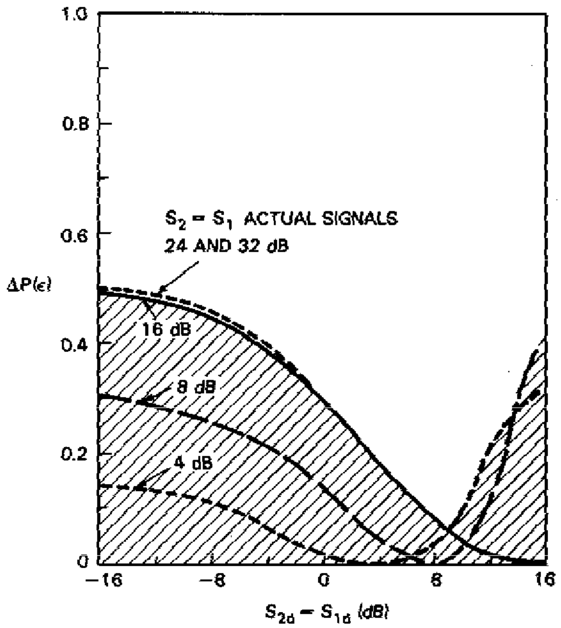


Fig. 4 — $\Delta P(\epsilon)$ vs design signal for $s_2 = s_1$ actual signals values, "assumed known-signal" design

Part (b) of statement (3), where the "assumed known signal" detector is used with $s_{2d} = s_{1d}$ and the actual signals take their minimum and maximum values, is analyzed by computing $\Delta P(\epsilon)$ for $s_{min} \rightarrow 0$ and $s_{max} \rightarrow \infty$. The reference "known-signal" decision rule leads to deciding H_2 whenever a large signal is present and H_0 otherwise. Thus, $P(\epsilon \mid \text{signal}) = \frac{1}{3}\,(0 + 1 + 0) = \frac{1}{3}$. Figure 5 is a plot of this condition. $P(\epsilon)$ has little variation about a value of $\frac{1}{2}$.

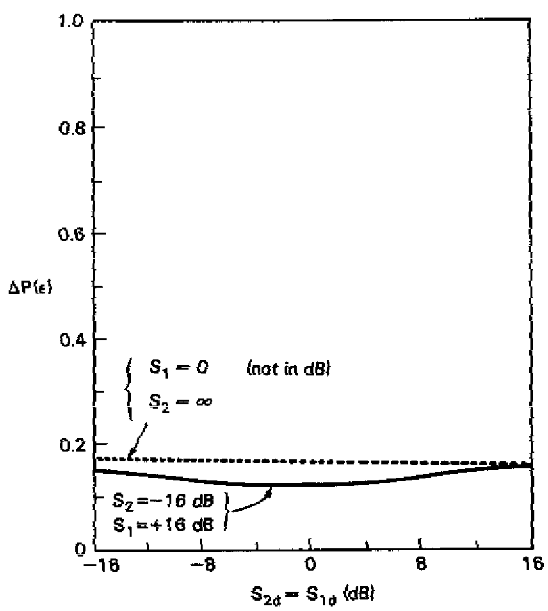


Fig. 5 — $\Delta P(\epsilon)$ vs design signal for min/max actual signal values, "assumed known-signal" design

Finally, part (c) of statement (3) combines the results of the maximum error $\Delta P(\epsilon)$ vs signal design found in parts (a) and (b). The combined curves from (a) and (b) are shown in Fig. 6. The best signal design is the design where the curve of the maximum values of $\Delta P(\epsilon)$ is a minimum. The best "assumed known-signal" design occurs for $s_{2d} = s_{1d}$ equal to 5 dB.

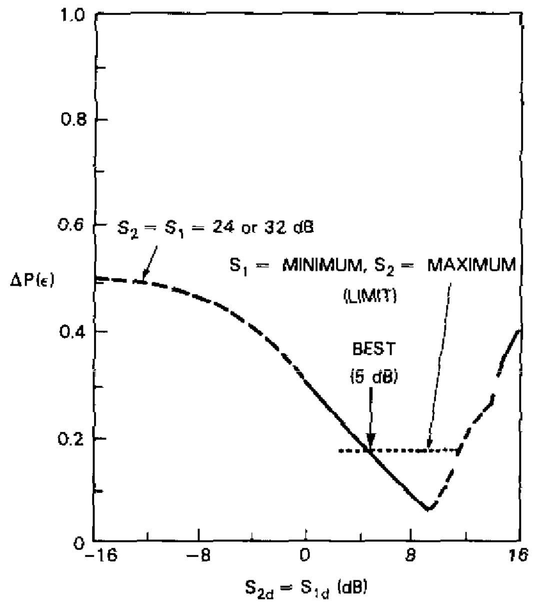


Fig. 6 — $\Delta P(\epsilon)$ vs design signal for best $s_2 = s_1$ and min/max actual signal values—"assumed known-signal" design

UNIFORMLY DISTRIBUTED SIGNAL DECISION RULE

Another approach to treating the unknown parameter problem is to assign a probability density function to these parameters and then to compute the error performance. We first develop the test for an arbitrary density function, then specialize to the uniform density case.

$p_{\vec{r}|\vec{s},H_n}(\vec{r}|\vec{s},H_n)$ = pdf of the received $\vec{r}$, conditioned on the random parameter $\vec{s}$ taking the value $\vec{s}$ and H_n being true,

$$p_{\vec{s}}(\vec{s}) = \text{pdf of } \vec{s},$$

$$p_{\vec{r}|H_n}(\vec{r}|H_n) = \int_{all\,\vec{s}} p_{\vec{r}|\vec{s},H_n}(\vec{r}|\vec{s},H_n)\, p_{\vec{s}}(\vec{s})\, d\vec{s}\,.$$

The decision rule for minimum error probability becomes (for equal $P(H_n)$):

$$\max_{H_n}\left[p_{\vec{r}|H_n}(\vec{r}|H_n)\right] \rightarrow \text{choose } H_n\,.$$

s_1 and s_2 are taken to be independent random quantities: $p_{\vec{s}}(s_1,s_2) = p_{s_1}(s_1)p_{s_2}(s_2)$. Assume s_1 and s_2 are uniformly distributed between s_l and s_u:

$$p_s(s) = \begin{cases} 1/(s_u - s_1)\,, & s_1 < s < s_u \\ 0\,, & elsewhere \end{cases}$$

$$p_{\vec{r}|\vec{s},H_0}(\vec{r}\,|\,\vec{s},H_0) = \frac{1}{2\pi}\, e^{-\frac{r_1^2 + r_2^2}{2}}$$

$$p_{\vec{r}|\vec{s},H_1}(\vec{r}\,|\,\vec{s},H_1) = \frac{1}{2}\,\frac{1}{2\pi}\, e^{-\frac{(r_1 - s_1)^2 + r_2^2}{2}} + \frac{1}{2}\,\frac{1}{2\pi}\, e^{-\frac{r_1^2 + (r_2 - s_2)^2}{2}}$$

$$p_{\vec{r}|\vec{s},H_2}(\vec{r}\,|\,\vec{s},H_2) = \frac{1}{2\pi}\, e^{-\frac{(r_1 - s_1)^2 + (r_2 - s_2)^2}{2}}$$

The indicated integration can be performed by evaluating two types of integrals.

$$I: \int_{s_l}^{s_u} \frac{1}{(2\pi)^{1/2}} e^{-\frac{r^2}{2}} \frac{1}{s_u - s_l} ds \quad = \quad \frac{1}{(2\pi)^{1/2}} e^{-\frac{r^2}{2}}$$

$$II: \int_{s_l}^{s_u} \frac{1}{(2\pi)^{1/2}} e^{-\frac{(r-s)^2}{2}} \frac{1}{s_u - s_l} ds$$

$$= \frac{1}{s_u - s_l} \int_{s_l}^{s^u} \frac{1}{(2\pi)^{1/2}} e^{-\frac{(r-s)^2}{2}} ds$$

$$= \frac{1}{s_u - s_l} \left[P(s_u - r) - P(s_l - r) \right]$$

$$\text{where } P(x) = \int_{-\infty}^{x} \frac{1}{(2\pi)^{1/2}} e^{-\frac{v^2}{2}} dv\,.$$

The conditional pdfs are then:

$$p_{\bar{r}|H_0}(\bar{r}|H_0) = \frac{1}{2\pi} e^{-\frac{r_1^2 + r_2^2}{2}},$$

$$p_{\bar{r}|H_1}(\bar{r}|H_1) = \frac{1}{2} \frac{1}{(2\pi)^{1/2}} \frac{1}{(s_u - s_l)} e^{-\frac{r_2}{2}} \left[P(s_u - r_1) - P(s_l - r_1) \right]$$

$$+ \frac{1}{2} \frac{1}{(2\pi)^{1/2}} \frac{1}{(s_u - s_l)} e^{-\frac{r_1}{2}} \left\{ P(s_u - r_2) - P(s_l - r_2) \right\},$$

and

$$p_{\bar{r}|H_2}(\bar{r}|H_2) = \frac{1}{(s_u - s_l)^2} \left[P(s_u - r_2) - P(s_l - r_2) \right] \left[P(s_u - r_1) - P(s_l - r_1) \right].$$

Define

$$W(r) = \frac{\sqrt{2\pi}}{s_u - s_l} e^{+\frac{r^2}{2}} \left\{ P(s_u - r) - P(s_l - r) \right\},$$

$$W_1 = W(r_1)\,,$$

and

$$W_2 = W(r_2)\,.$$

Note that W_1 and W_2 are likelihood ratios of H_1 and H_2 respectively. The pdfs become:

$$p_{\bar{r}|H_0}(\bar{r}|H_0) = \frac{1}{2\pi} e^{-\frac{r_1^2 + r_2^2}{2}} [1],$$

$$p_{\bar{r}|H_1}(\bar{r}|H_1) = \frac{1}{2\pi} e^{-\frac{r_1^2 + r_2^2}{2}} \left[\frac{W_1 + W_2}{2} \right], \text{ and}$$

$$p_{\bar{r}|H_2}(\bar{r}|H_2) = \frac{1}{2\pi} e^{-\frac{r_1^2 + r_2^2}{2}} \left\{ W_1 W_2 \right\}.$$

The tests are expressed in terms of W_1 and W_2 by

$$\max_{H_n} \left[\overset{H_0}{\overset{|}{1}}, \quad \overset{H_1}{\overset{|}{1/2(W_1+W_2)}}, \quad \overset{H_2}{\overset{|}{W_1+W_2}} \right] \rightarrow \text{choose } H_n .$$

This is of the same form as for the known-signal case where $\{W_n\}$ replaces $\{R_n\}$. *Consequently, the* decision boundaries are already known and are shown in Fig. 7. Thus, if no unknown parameters are present and the signal and/or noise are independent of each other, the decision boundaries can be written as in Fig. 7 where the axes are labeled "likelihood ratio," Fig. 7 holds for all likelihood ratios.

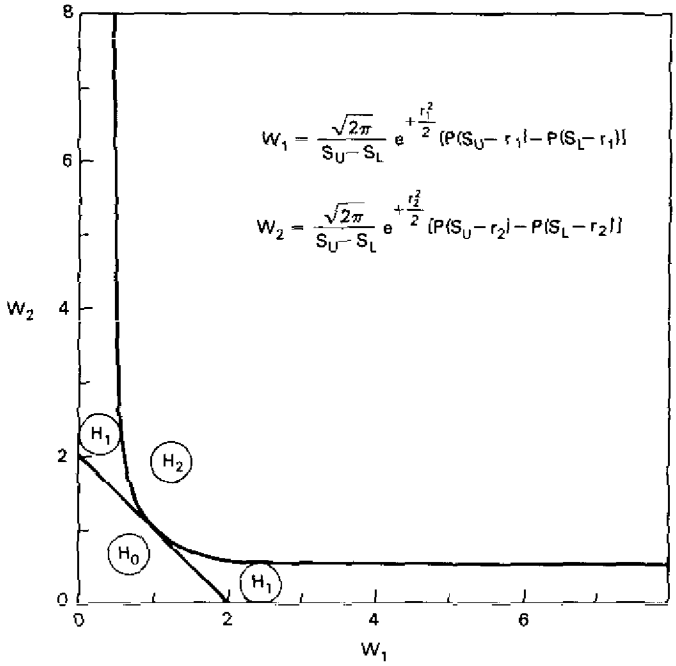


Fig. 7 — "Uniformly distributed signals" decision regions

The signal location considered is a square region, centered at the origin of the signal space. It is defined by $s_L = -s_U$, where s_L = lower limit and s_U = upper limit of signals as shown in Fig. 8. The performance measure $\Delta P(\epsilon)$ is computed, and the value and location of $\Delta P(\epsilon)_{MAX}$ are given in Table 4. From Table 4, the best design occurs around s_U = 14 dB. The maximum values of the actual signals shown in Table 4 are plotted in Fig. 9 for each design s_U near the solution, and the true minimax solution is s_U = 13 dB. This information is shown graphically to allow interpolation in finding the minimax solution. Table 5 shows $\Delta P(\epsilon)$ vs all actual signals for the best design s_U = 13 dB.

Fig. 8 — *Uniformly distributed signal region*

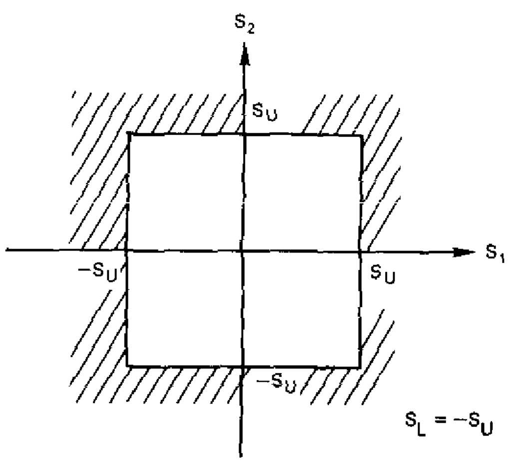


Table 4 — Value and Location of Largest $\Delta P(\epsilon)$ — "Uniformly Distributed" Detector

S_U (dB)	$\Delta P(\epsilon)$	Location of target
8	0.367	Maximum/maximum
10	0.240	Maximum/maximum
12	0.175	Maximum/maximum
14	0.165	Maximum/minimum
16	0.173	Maximum/interior
18	0.183	Maximum/interior
20	0.183	Interior/interior
22	0.183	Interior/interior

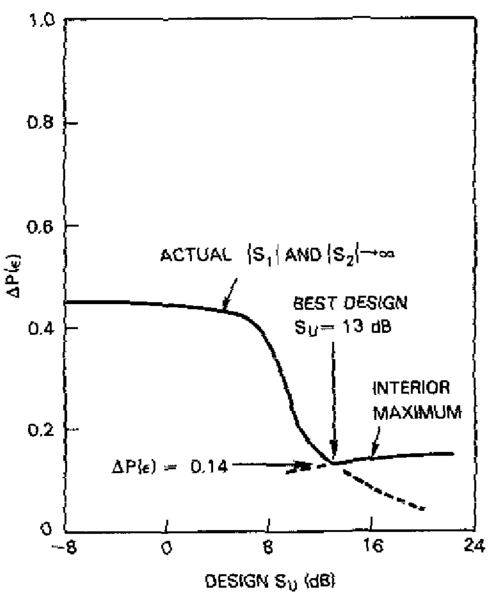


Fig. 9 — Determining best "uniformly distributed signals" design

Table 5 — $\Delta P(\epsilon)$ for "Uniformly Distributed Signals" Detector
(at best; $S_U = -S_L = 13$ dB)

	-16	-14	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12	14	16
-16	0.0292	0.0357	0.0440	0.0472	0.0603	0.0672	0.0807	0.0927	0.1117	0.1222	0.1305	0.1383	0.1288	0.1307	0.1338	0.1415	0.1453
-14	0.0358	0.0362	0.0435	0.0530	0.0550	0.0705	0.0800	0.0918	0.1113	0.1207	0.1287	0.1377	0.1253	0.1275	0.1323	0.1380	0.1417
-12	0.0430	0.0483	0.0445	0.0547	0.0645	0.0732	0.0828	0.0958	0.1135	0.1227	0.1277	0.1352	0.1237	0.1273	0.1297	0.1362	0.1402
-10	9.0497	0.0552	0.0588	0.0578	0.0663	0.0802	0.0883	0.1027	0.1107	0.1212	0.1282	0.1352	0.1237	0.1215	0.1220	0.1305	0.1338
-8	0.0585	0.0622	0.0673	0.0727	0.0768	0.0873	0.0962	0.1085	0.1128	0.1220	0.1227	0.1345	0.1213	0.1192	0.1215	0.1280	0.1310
-6	0.0655	0.0728	0.0767	0.0802	0.0810	0.0890	0.1015	0.1060	0.1173	0.1295	0.1298	0.1308	0.1172	0.1180	0.1173	0.1220	0.1268
-4	0.0813	0.0848	0.0848	0.0938	0.1005	0.1013	0.1075	0.1172	0.1322	0.1368	0.1375	0.1330	0.1212	0.1140	0.1173	0.1222	0.1260
-2	0.0933	0.0927	0.0977	0.1072	0.1132	0.1178	0.1185	0.1328	0.1417	0.1443	0.1417	0.1340	0.1277	0.1160	0.1185	0.1238	0.1270
0	0.1057	0.1107	0.1137	0.1165	0.1163	0.1240	0.1280	0.1380	0.1457	0.1482	0.1432	0.1368	0.1220	0.1107	0.1163	0.1233	0.1262
2	0.1188	0.1182	0.1197	0.1180	0.1290	0.1282	0.1333	0.1420	0.1512	**0.1532**	0.1380	0.1340	0.1148	0.1070	0.1057	0.1132	0.1160
4	0.1280	0.1292	0.1302	0.1322	0.1365	0.1308	0.1372	0.1430	0.1473	0.1465	0.1257	0.1173	0.1015	0.0895	0.0930	0.1018	0.1043
6	0.1308	0.1310	0.1278	0.1250	0.1303	0.1263	0.1250	0.1292	0.1360	0.1293	0.1082	0.1013	0.0840	0.0822	0.0853	0.0938	0.0975
8	0.1308	0.1270	0.1250	0.1230	0.1195	0.1177	0.1165	0.1150	0.1122	0.1058	0.0848	0.0805	0.0670	0.0610	0.0680	0.0783	0.0830
10	0.1247	0.1240	0.1240	0.1207	0.1202	0.1152	0.1108	0.1092	0.1032	0.0970	0.0803	0.0720	0.0608	0.0637	0.0748	0.0872	0.0925
12	0.1308	0.1295	0.1282	0.1258	0.1243	0.1207	0.1135	0.1107	0.1112	0.1035	0.0855	0.0838	0.0735	0.0787	0.0910	0.1040	0.1090
14	0.1372	0.1355	0.1348	0.1325	0.1283	0.1238	0.1148	0.1172	0.1172	0.1110	0.0913	0.0925	0.0842	0.0897	0.1035	0.1165	0.1217
16	0.1417	0.1393	0.1393	0.1368	0.1322	0.1277	0.1188	0.1210	0.1217	0.1153	0.0962	0.0963	0.0892	0.0953	0.1098	0.1230	0.1285

SEQUENTIALLY APPLIED NEYMAN-PEARSON DECISION RULES

The lack of knowledge about the signal levels s_1 and s_2 suggest formulating a test that does not use these signal levels. The Neyman-Pearson test comes close to meeting this condition. This test sets the threshold when the signal is absent so that a given probability of false alarm (P_{fa}) occurs. The probability of missed detection is computed from knowing this threshold and the signal-to-noise ratio.

Three hypotheses, H_0, H_1, and H_2 are present in this problem while the standard Neyman-Pearson technique applies to a binary problem. To apply the Neyman-Pearson technique, the problem is expressed as a series of binary tests and the Neyman-Pearson technique is applied to each one in sequence. The Neyman-Pearson technique's parameter P_{fa} sets a threshold for a single signal. The symmetry of the problem leads to applying the same P_{fa} for both s_1 and s_2. Then, both r_1 and r_2 are tested to see if either s_1 or s_2 are present by comparing r_1 and r_2 to this threshold. If both do not cross the threshold, then H_0 is declared and the test is stopped. An additional pair of tests is made when H_0 is not declared. The number of signals declared present determines which H_n is chosen.

The threshold for the test uses the noise-only hypothesis; thus pdf can be written

$$p(n) = \frac{1}{(2\pi)^{1/2}} e^{-1/2n^2}$$

$$1/2P_{fa} = \int_{\tau_0}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{-1/2n^2} dn .$$

Figure 10 shows τ_0 vs P_{fa} for Gaussian noise, and Fig. 11 shows the decision regions. The combined test applied to the received signal $\overline{r}^*$ is then:

If: $|r_1^*| < \tau_0$, and $|r_2^*| < \tau_0$
Then: declare H_0
Else:
  If: $|r_1^*| > \tau_0$, and $|r_2^*| > \tau_0$
  Then: declare H_2
  Else: declare H_1
End.

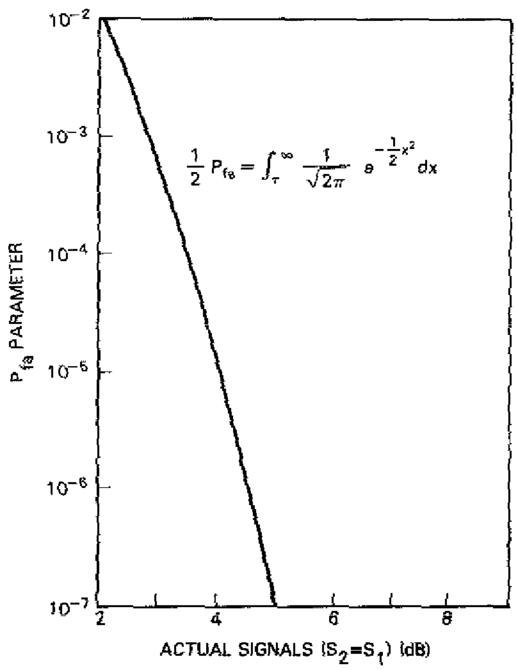


Fig. 10 — Threshold τ

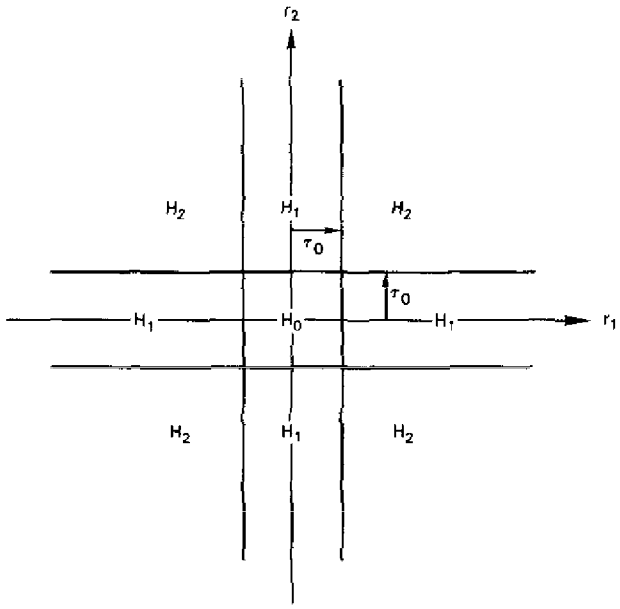


Fig. 11 — Neyman-Pearson decision region

Figure 12 is a plot of $P(\epsilon)$ vs signal level $(s_2 = s_1)$ for various P_{fa}. The error probability approaches a limit slightly less than P_{fa} as the signal strength increases. Since the "known signal" detector has $P_{fa} \rightarrow 0$ for large signals, a design using small P_{fa} seems desirable. Figure 12 shows that the small signal performance is poor for small P_{fa} designs and large signal performance is poor for large P_{fa} designs. A compromise must be made to obtain good performance at all signal levels.

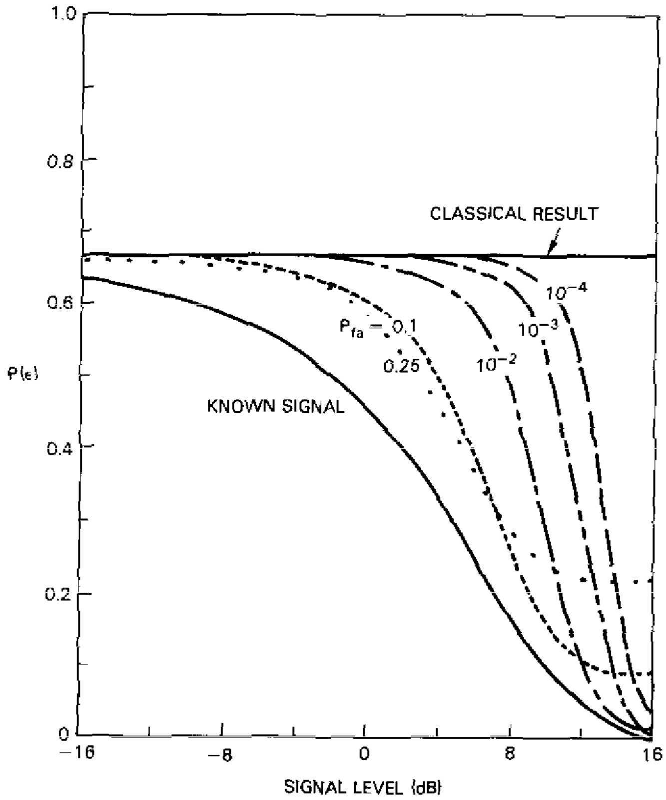


Fig. 12 — Error probabilities for Neyman-Pearson case, uncorrelated noise

A particular design value for the probability of false alarm is determined by using a minimax criteria. In this case, the worst differential error $\Delta P(\epsilon)$ over all actual signals is plotted vs P_{fa} (Fig. 13). The best design for P_{fa} is $P_{fa} = 0.08$, because the worst $\Delta P(\epsilon)$ is a minimum at this point. The actual signal levels that yield the worst $\Delta P(\epsilon)$ are noted on the figure. The solid line indicates the worst $\Delta P(\epsilon)$ is occurring where $s_2 = s_1$ for these various P_{fa}. The dotted line indicates the worst $\Delta P(\epsilon)$ occurs for $s_2 \neq s_1$.

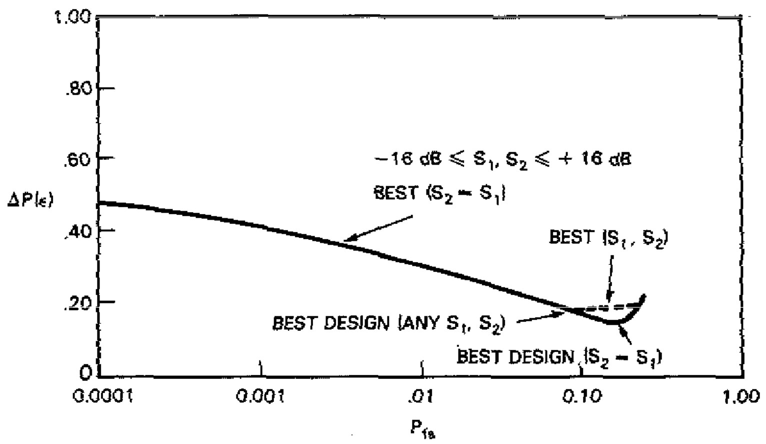


Fig. 13 — ΔP (error) vs P_{fa}—Neyman-Pearson uncorrelated noise

The remainder of this report investigates the effect of noise correlation on the performance of some Neyman-Pearson detectors.

CORRELATED NOISE

Decision making in correlated noise is made by generalizing the Neyman-Pearson work. The hypothesis of no signals being present, H_0, is tested by decorrelating the jointly Gaussian noise and applying a Neyman-Pearson P_{fa} test to both r_1 and r_2. Jointly Gaussian random variables can be made independent of each other by applying the proper linear transformation. The parameter θ in the transformation is selected so that the transformed random variables are independent of each other.

The pdf of the noise (for $\sigma_2 = \sigma_1 = 1$) is:

$$p(n_1, n_2) = \frac{1}{2\pi(1-\rho^2)^{1/2}}\, e^{-\frac{n_1^2 - 2\rho n_1 n_2 + n_2^2}{2(1-\rho^2)}}$$

The transformation used is:

$$m_1 = n_1 \cos\theta + n_2 \sin\theta$$

$$m_2 = n_1 \sin\theta + n_2 \cos\theta\,.$$

The noise is decorrelated by $\theta = \frac{1}{2}\tan^{-1}\left[\frac{2\rho\sigma_1\sigma_2}{\sigma_2^2 - \sigma_1^2}\right]$.

In this case, $\sigma_2 = \sigma_1$ so that $\theta = \frac{\pi}{4}$ and $\sin\theta = \cos\theta = 1/\sqrt{2}$. The uncorrelated Gaussian density of the new variables is given by

$$p_{\vec{m}}(\vec{m}) = \frac{1}{2\pi\sigma_{m_1}\sigma_{m_2}}\, e^{-\frac{1}{2}\left[\frac{m_1^2}{\sigma_{m_1}^2} + \frac{m_2^2}{\sigma_{m_2}^2}\right]},$$

where $\sigma_{m_1}^2 = 1 + \rho$ and $\sigma_{m_2}^2 = 1 - \rho$.

The m_1 and m_2 thresholds are obtained by causing the threshold τ_0 to apply to meet the design parameter P_{fa} with $\sigma = 1$ and then adjusting this threshold for $\sigma \neq 1$:

$$\frac{1}{2} P_{fa} = \int_{\tau_0}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{-1/2n^2} dn .$$

For nonunity standard deviation $x = n\ \sigma$,

$$\frac{1}{2} P_{fa} = \int_{(\sigma\tau_0)}^{\infty} \frac{1}{(2\pi)^{1/2}\sigma}\ e^{-1/2(x/\sigma)^2} dx ,$$

$$\tau_1 = \sigma_{m_1} \tau_0 = \sqrt{1+\rho}\,\tau_0 , \text{ and}$$

$$\tau_2 = \sigma_{m_2}\tau_0 = \sqrt{1-\rho}\,\tau_0 .$$

The test for H_0 is

If: $|\tilde{r}_1^*| < \tau_1$ and $|\tilde{r}_2^*| < \tau_2$
Then: declare H_0,
Else: test for H_1 or H_2,
End.

where $\tilde{r}_1^*$ and $\tilde{r}_2^*$ are the transformed received signals r_1^* and r_2^*.

If H_0 is declared true, then the test is completed; if H_0 is not declared, the test for H_1 and H_2 is conducted by using two additional tests. The first test is to determine if s_1 is present conditioned on s_2 being present, and the second tests for s_2 given s_1. To conduct these tests, two methods are investigated. The first test is the same as the test used for uncorrelated noise, and the second is new. The first method ignores correlation in all tests made; its development has already been discussed. The second method conducts tests based on conditional densities. The densities are conditioned on one signal being zero and the other being an estimate. The estimate is "most likely."

The results are obtained by using the Neyman-Pearson method developed in the previous section for $P_{fa} = 0.0001$, 0.001, 0.01, 0.1, and 0.25, and for noise correlation coefficient $\rho = 0.99$. The other two methods considered are similarly plotted. The following points can be made from Fig. 14.

- The performance of small signals approaches the classical result of $P(\epsilon) = 2/3$, which is reasonable since no procedure can do better for zero-strength signals.

- As the signal increases, the performance breaks away from the classical result. Consider, for example, the signal levels encountered for $P(\epsilon) = 0.6$ (90% of the classical value). The signal at $P_{fa} = 0.0001$ has to be 15 dB stronger than the $P_{fa} = 0.25$ value. The 15 dB signal difference decreases to 7 dB when $P(\epsilon) = 0.3$ due to the greater slope of the curves at small P_{fa} values.

- Signals lying between -10 dB and $+7$ dB have error probabilities that are better for large P_{fa} values.

- The large signal performance gets poorer for larger P_{fa} values. For example, the $p_{fa} = 0.25$ curve never gets below 0.165. This poor performance is due to nonzero $P(\epsilon)$ for H_0 and H_1.

A small P_{fa} design gives poor performance at moderate signals but good performance at large signals, and a large P_{fa} design gives poor performance at large signals but good performance at midlevel signals; a nominal value of $p_{fa} = 0.1$ was chosen for further analysis.

Figure 15 is a plot of $P(\epsilon)$ vs actual signal strength $(s_2 = s_1)$ for $P_{fa} = 0.1$ and $\rho = 0.9$, 0.99, and 0.999. These ρ values are both smaller and larger than the nominal ρ used in the curves of Fig. 14. Figure 15 shows that there is only slight difference for all values of ρ, the most obvious being the strong signal performance. Here, the performance at the smaller ρ values is best and is ~35% lower than the 0.075 error performance at $\rho = 0.999$.

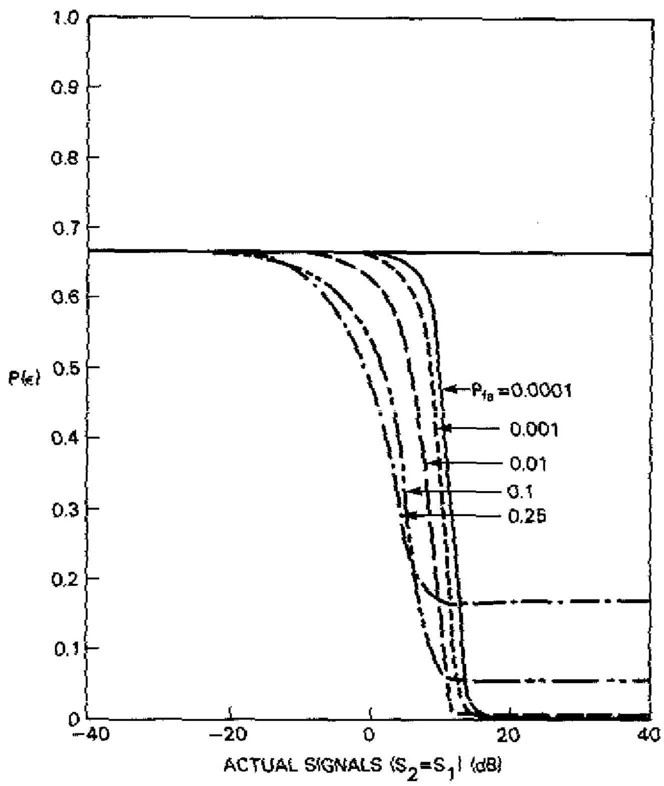


Fig. 14 — Ignore correlation—ϵ = 0.99

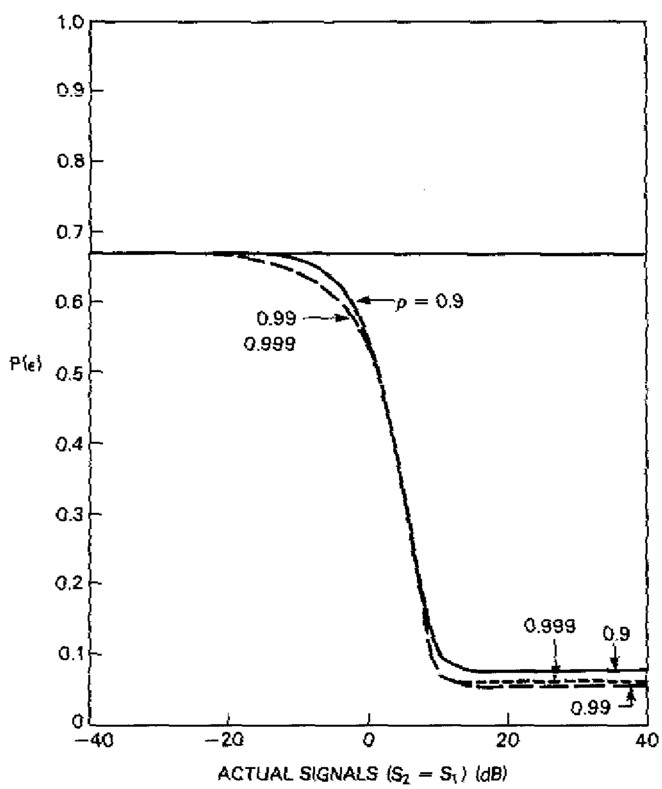


Fig. 15 — Ignore correlation—P_ϵ = 0.1

Most Likely Estimate (Correlated Noise)

Conducting a Neyman-Pearson test for the presence of a signal requires estimating unknown parameters and integrating the pdf, with this signal not present, over a region set by P_{fa} considerations. The second method of accounting for correlated noise estimates the unknown signal s_2 in $p(r_1|r_2 = r_2^*, s_1 = 0, s_2)$ by maximizing the joint pdf $p(r_1, r_2)$.

$$p(r_1,r_2) = \frac{1}{2\pi(1-\rho^2)} e^{-\frac{1}{2(1-\rho^2)^{1/2}}[(r_1 - s_1)^2 - 2\rho(r_1 - s_1)(r_2 - s_2) + (r_2 - s_2)^2]}$$

where r_1 and r_2 equal their received values and $s_1 = 0$. By setting

$$\frac{\partial p(r_1,r_2)}{\partial s_2} = 0,$$

we get $\hat{s}_2 = r_2^* - \rho r_1^*$ and

$$p(r_1|r_2) = \frac{p(r_1,r_2)}{p(r_2)}$$

$$= \frac{1}{(2\pi)^{1/2}(1-\rho^2)^{1/2}} e^{-\frac{(r_1-\rho^2 r_1^*)^2}{2(1-\rho^2)^{1/2}}}$$

$$= N\ (\rho^2 r_1^*, \sigma^2 = 1-\rho^2)\,.$$

For a Gaussian random variable r_1, the desired threshold τ, centered about the mean $\rho^2 r_1^*$, is $\tau = \sqrt{1-\rho^2}\tau_0$ where τ_0 is the threshold for $\rho = 0$. The random variable $(r_1 - \rho^2 r_1^*)$ is a Gaussian random variable with zero mean and variance $= 1 - \rho^2$. The rule is to declare s_1 absent if $|r_1-\rho^2 r_1^*| < \tau$. Since $r_1 = r_1^*$ when the test is applied,

$$|r_1^*|(1-\rho^2) < \tau \rightarrow s_1 \text{ absent}$$

$$|r_1^*| < \frac{\tau_0}{(1-\rho^2)^{1/2}} \rightarrow s_1 \text{ absent, else } s_1 \text{ declared present}.$$

A similar test holds for declaring s_2 present or absent. The declaration of H_1 or H_2 is made as before, s_1 and s_2 present $\rightarrow H_2$. otherwise H_1.

The error performance obtained by estimating the unknown signals by maximizing the joint density function ("most likely" method) is given in Figs. 16 and 17. Figure 16 is a plot of $P(\epsilon)$ vs actual signal strength ($s_2 = s_1$) for $\rho = 0.99$ and values of P_{fa} from 0.0001 to 0.25. This set of curves shows a plateau not found for the other estimation methods. The curve of $P(\epsilon)$ vs s_2 $(=s_1)$ decreases from the classical limit of 2/3 to a level that depends on P_{fa} (signal levels from 0 to 15 dB). For very weak signals, larger values of p_{fa} give better small signal performance. The plateau extends from signal values of ~ -5 dB to $+25$ dB, the actual limits depend on P_{fa}. For strong signals, the error $P(\epsilon)$ is smallest for the smallest P_{fa}. The large signal error performance is determined only by $P(\epsilon|H_0)$ (since $P(\epsilon|H_1)$ and $P(\epsilon|H_2) = 0$), and is given by

$$P(\epsilon) = P(H_0)P(\epsilon|H_0) = \frac{1}{3}\left[1 - (1-P_{fa})^2\right].$$

For $P_{fa} = 0.25$, this equation gives $P(\epsilon) = 0.146$, which agrees with the value computed. This does not verify the accuracy of the Monte Carlo method used because the Monte Carlo method gives exact values for $P(\epsilon|H_1)$ and $P(\epsilon|H_2)(= 0)$ that are accurate to an infinite number of places. An exact match would not occur for smaller signals. There is no overall better design because the P_{fa} values giving the lowest $P(\epsilon)$ changes from (starting at small signals and progressing to larger signals): $P_{fa\max} \rightarrow P_{fa\min} \rightarrow P_{fa\max} \rightarrow P_{fa\min}$.

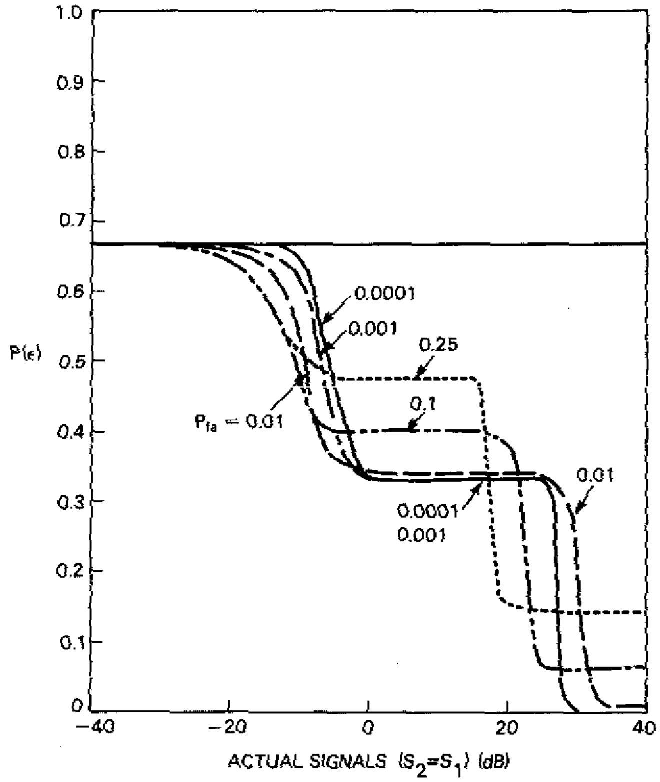


Fig. 16 — "Most likely" signal estimation, ρ = 0.99

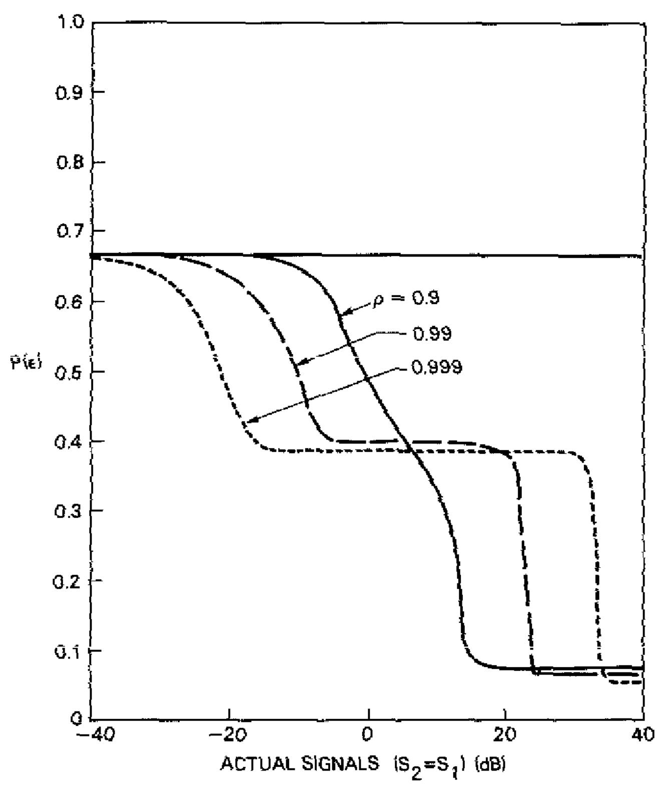


Fig. 17 — "Most likely" signal estimation, P_{fa} = 0.1

Figure 17 plots $P(\epsilon)$ vs actual signal $(s_2 = s_1)$ for $P_{fa} = 0.1$ and $\rho = 0.9$, 0.99 and 0.999 for the "most likely" signal estimation method. These curves show that the plateau length increases with increasing ρ. The $\rho = 0.9$ case almost causes all indications of this plateau to disappear. The drop in $P(\epsilon)$ of the small signal side of the plateau is caused by $P(\epsilon|H_1)$ going from a value near one to zero, while the drop for signals to the right of the plateau drop because the conditional error probability $P(\epsilon|H_2)$ goes from one to zero. The remaining strong signal $P(\epsilon)$ is determined by $P(\epsilon|H_0)$. Large ρ values markedly improve the small signal performance but also markedly reduce the moderate-to-large signal performance. For large signals, large ρ gives the best performance.

Error performance of the two methods are compared in Fig. 18, a plot of $P(\epsilon)$ vs actual signal $(s_2 = s_1)$ and for $\rho = 0.99$ and $P_{fa} = 0.1$. Small signal error performance is best for the "most likely" estimation method and continues to be best until the signals reach +5 dB. Then the "ignore correlation" method does markedly better until the signals reach +25 dB and remains marginally better for larger signals. All methods have signal ranges where they perform noticeably worse than the best, thus no method is chosen "best." If forced to choose, the choice would be for the method that ignores correlation. This is because it has fractionally less degradation where it is not the best compared to the "most likely" method.

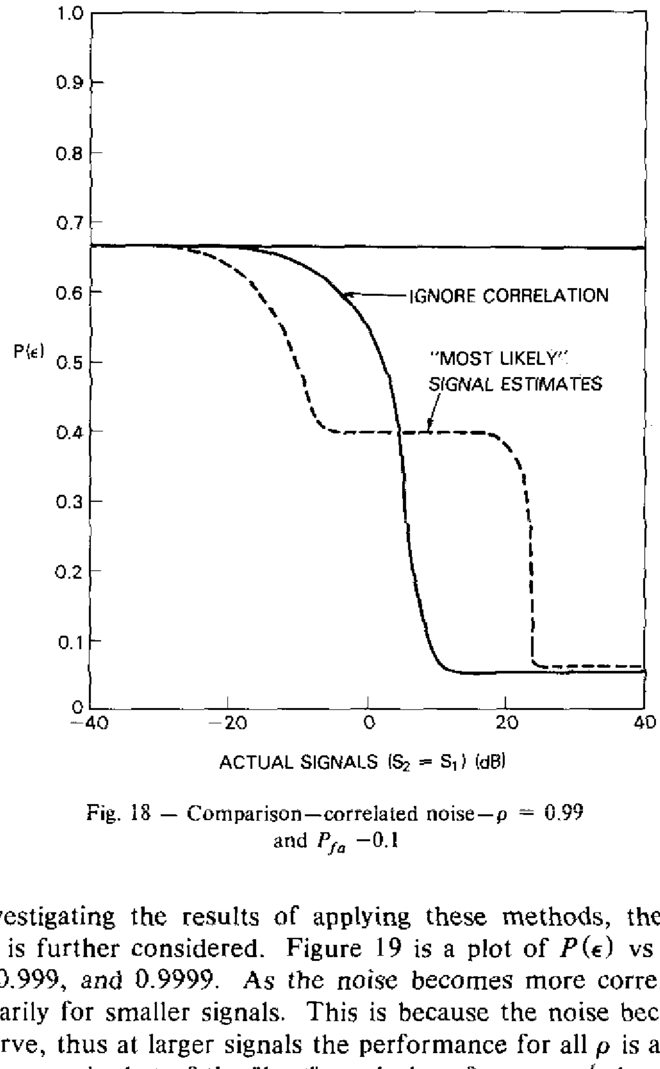


Fig. 18 — Comparison—correlated noise—$\rho = 0.99$ and P_{fa} −0.1

In addition to investigating the results of applying these methods, the effect of the reference "known signal" detector is further considered. Figure 19 is a plot of $P(\epsilon)$ vs signal strength $(s_2 = s_1)$ for ρ = 0., 0.9, 0.99, 0.999, and 0.9999. As the noise becomes more correlated ($\rho \rightarrow 1$) the performance gets better, primarily for smaller signals. This is because the noise becomes less uncertain. All curves join the lower curve, thus at larger signals the performance for all ρ is almost the same. At very small signals, the performance is that of the "best" *a priori* performance (where the received signals are not considered in making a decision). Between these extremes, the ρ-dependency can be seen. In the limit $\rho \rightarrow 1$, the noise voltages are equal; thus under H_0, no errors are made if the rule "Declare H_0

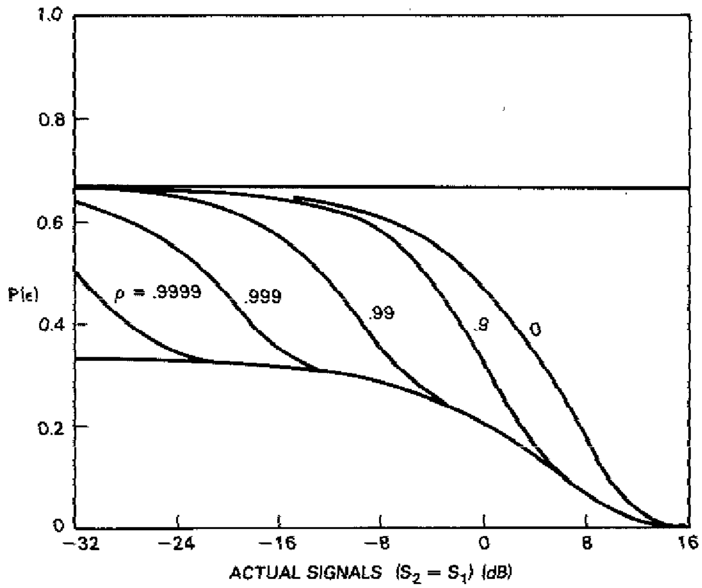


Fig. 19 — Effects of correlation and known signals

present if and only if $r_2^* = r_1^*$" is used. For very small signals, no noticeable difference is present between H_1 and H_2 so an error is made half of the time, i.e., $P(\epsilon|H_1) + P(\epsilon|H_2) = 1/3$. Hence, $P(\epsilon) = 1/3$. The general shape of the performance curves are about the same for all ρ, and the curves are shifted ~10 dB for each "9" change in ρ (or for each 10 dB reduction in $(1-\rho)$).

COMPARISONS

The error probability of the "best" of each decision rule considered for the uncorrelated cases are plotted in Fig. 20. The known signal case is a lower bound (as expected). The best "assumed known signal" design is for $s_{2d} = s_{1d} = 8$ dB, the "uniformly distributed" design is best for $s_u = 13$ dB, and the Neyman-Pearson design is best for $P_{fa} = 0.08$; each gives almost the minimum $\Delta P(\epsilon)$ within that particular family of designs.

For small signals, the "best" (lowest) error performance is obtained for the "assumed known signal" detector method. The uniformly distributed Neyman-Pearson are always worse than the "assumed known signal" case until the signal levels are near 10 dB where (by coincidence) they all come near each other. For larger signals, the lowest $P(\epsilon)$ design becomes the Neyman-Pearson. The "uniformly distributed" detector is worse than the "assumed known signal" detector except for large signals. No rule is uniformly better [2, p. 796] than others, so choosing the best rule requires judging the advantages and the disadvantages of each. My choice of those shown is the "assumed known signal" because it is close to the "known signal" detector error probability results for small signals.

Correlated noise introduces another piece of information (correlation coefficient) into the problem. The Neyman-Pearson tests required an estimate of the (unknown) signals, not set equal to zero as before, in the conditional pdf. The selected method is "ignore correlation."

CONCLUSIONS

The error performance in determining the number of signals present when each signal (if present) combines linearly with Gaussian noise and with only a single look-per-signal (if available) is compared and evaluated numerically; the one giving the lowest error probability (or close, since none were uniformly best) is selected. The classical method estimates any needed unknown parameters and places their estimates in the probability density functions. The hypothesis with the largest probability density